Theory of the Attenuation of Very High Amplitude Sound Waves

by

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SUMMARY

The propagation of continuous plane progressive sound waves whose pressure variation is of the order of one tenth of the average pressure is discussed. It is shown that regardless of the initial wave form shocks will develop at the leading front of each wave after several wave lengths of propagation. Assuming the resulting stable wave form to be saw-tooth in character the attenuation of these repeated shocks is derived from shock wave theory. Writing $p_2/p_1 = 1 + \delta$, where $p_2 - p_1$ is the pressure discontinuity at the shock, it is shown that

$$\frac{1}{\delta} - \frac{1}{\delta_0} = \frac{\gamma+1}{2\gamma} \cdot \frac{X - X_0}{\lambda} ,$$

where δ_0 is the value of δ at the distance, X, equal to X_0, γ is the ratio of specific heats and λ is the wave length of the sound. This result is discussed and compared with previously published studies of the attenuation of single N waves and found to be compatible. Also it is shown that Fay's solution of the hydrodynamic equations including the effects of viscosity, which shows the stable wave form to be a saw-tooth, may be extended to yield the attenuation rate derived here.

The Attenuation of Very High Amplitude Sound Waves

I. Introduction

The problem of concern here is the attenuation experienced
by a very high amplitude progressive, plane sound wave — one whose
peak to trough pressure variation is of the order of one tenth of the
mean pressure. Consider then a wave whose pressure variation ini-
tially can be described by a sine function. As is well known[*], the
wave will be distorted as it propagates since the peaks travel at a
higher velocity than the troughs and consequently there will be a ten-
dency for the gradient to increase on the leading front of each wave
and decrease on the trailing front. Fay[2] has shown that a limiting
form of the wave when one includes viscosity as an attenuative process
is saw-tooth with the leading front approaching a vertical slope. It
would be possible, by means of an integration of the hydrodynamic
equations[3], to trace out in detail the process of this deformation up
to the development of the shock front, but, for the purposes of our
discussion, all that is needed is a treatment which will yield the ap-
proximate distance of propagation necessary for this deformation to
become complete. To obtain this it is necessary to recognize that
the instantaneous velocity of propagation of a given part of the wave is
equal to the sum of the sound velocity at the given phase of the wave,

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[*]See, e. g., pp. 30-41 of Ref. 1.

$\underline{c}$, and the local particle velocity, $\underline{u}$. Thus, in Figure (1),
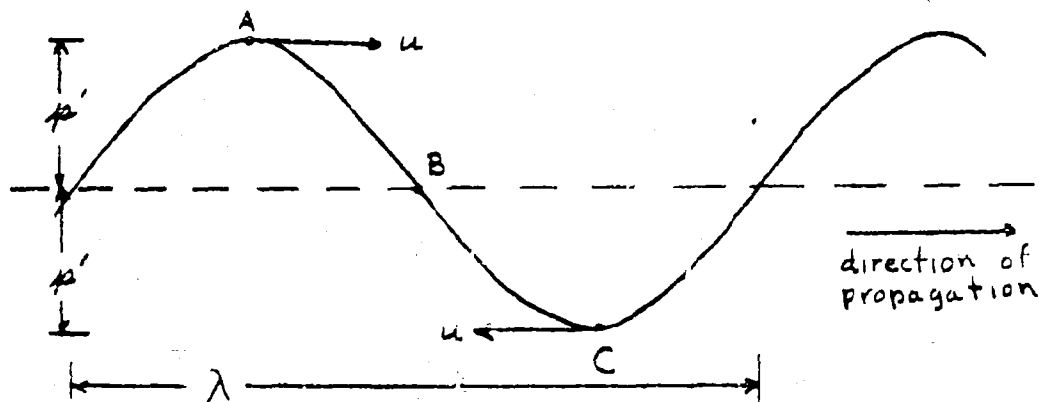

Fig. 1.

which represents a sine wave, the point marked A will travel faster

than B because the temperature at A is elevated and consequently

$c_A > c_B$, and because the particle velocity, $\underline{u}$, in the direction of prop-

agation is greater than that at B. For an adiabatic wave,

$$c + u = c_B + \frac{\gamma + 1}{2} \cdot u ,$$

where γ is the ratio of specific heats.

A sufficient approximation (for the present purposes) will be

to assume that the relation between $\underline{p'}$ and $\underline{u}$ is that which obtains for

infinitesimal sound waves, and that $\underline{u}$ at the peak remains constant as

it distorts. With this approximation it is possible to calculate the

distance of propagation necessary before point A overtakes point B.

If c_0 is the sound velocity at point B, ρ_0 the density of gas at point

B, then this distance, X, will be given by

$$\frac{\gamma+1}{2} \cdot \frac{p'}{\rho_0 c_0} \cdot \frac{x}{c_0} = \frac{\lambda}{.4},$$

where the relation $u = p'/\rho_0 c$ has been used.

Thus,
$$\frac{x}{\lambda} = \frac{1}{2(\gamma+1)} \cdot \frac{\rho_0 c_0^2}{p'}.$$

Using $c_0^2 = \gamma p_0/\rho_0$, where p_0 is the pressure at point B, one obtains

$$\frac{x}{\lambda} = \frac{\gamma}{2(\gamma+1)} \cdot \frac{1}{p'/p_0}.$$

Thus, for $p'/p_0 = 0.05$ and $\gamma = 1.4$,

$$\frac{x}{\lambda} = 5.8.$$

It is thus clear that for excess pressure amplitudes, p', equal to 0.05 of the ambient atmospheric pressure, it is reasonable to expect shock wave character, at the very most, several wave-lengths from the source, and, moreover, this result is correct as to order of magnitude regardless of the form of the initially produced wave. Consequently, of prime importance in considering the attenuation of very high amplitude sound waves is the attenuation of repeated shock waves, and it is to this problem which attention will be restricted.

There are two ways of approaching the problem. The more fundamental approach is typified by the work of Fay[2], and involves a

solution of the hydrodynamical equations including the attenuative process of viscosity which, as pointed out, limits the maximum pressure gradient. (Without such attenuation one is led to multiple valued solutions after the wave crest overtakes the trough.) In this fashion Fay was able to show that a "stable" wave form is approached which is saw-tooth. A second procedure (one developed in detail in this paper) which has proved to be lucrative, can be used and is characterized by the application of the Rankine-Hugoniot shock relations. One starts out by assuming the wave has a saw-tooth character and the leading edge is characterized by a discontinuity in pressure, density, particle velocity and temperature. Application of the shock relations leads to the result that there is an entropy increase across the discontinuity, or shock, and this entropy increase can be interpreted as a space rate of decrease of amplitude. It is to be noted that this procedure leads to a calculation of attenuation rates without specifically mentioning any attenuative processes. (The substitute for an explicit formulation of these processes is contained in the assumption of the presence and stability of the discontinuity.) It will be shown in a later section of this paper that the attenuation obtained by application of the shock relations may also be obtained from a result given by Fay[2].

The development given here will be a first order theory in

the sense that all but the lowest orders of significant terms will be dropped. Also the waves, it will be assumed, are exclusively plane progressive waves.

II. Theory

A. Propagation of a Discontinuity.

The results which will be obtained in this section A can be found in many texts (see e. g., Refs. (3) and (5)) but, for the sake of completeness, a brief outline of the procedure will be given.

Consider a plane discontinuity moving to the right with a velocity U. The subscript 2 refers to the medium to the left of the discontinuity and 1 to the medium to the right of the discontinuity.

$$p_2, \ p_1 = \text{pressure}$$

$$\rho_2, \ \rho_1 = \text{density}$$

$$S_2, \ S_1 = \text{specific entropy}$$

$$u_2, \ u_1 = \text{particle velocity}$$

$$T_2, \ T_1 = \text{temperature}$$

$$e_2, \ e_1 = \text{specific internal energy}$$

It will be assumed that the gas through which the discontinuity moves is a polytropic gas, i. e., $p/\rho = RT$ and $e = C_v T$, where C_v is the specific heat at constant volume.

Suppose a velocity of flow, $-U$, is superposed on the whole

gas. Then the discontinuity will remain fixed; gas will enter the dis-
continuity from the right with a velocity $v_1 = U - u_1$ and leave with a
velocity $v_2 = U - u_2$.

The three shock relations assert that (1) mass, (2) momen-
tum, and (3) energy are conserved for the motion. They lead respec-
tively to the three relations,

$$\rho_1 v_1 = \rho_2 v_2 \tag{1}$$

$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2 \tag{2}$$

$$\frac{p_1}{\rho_1} + \frac{1}{2} v_1^2 + e_1 = \frac{p_2}{\rho_2} + \frac{1}{2} v_2^2 + e_2 \tag{3}$$

v_1, v_2, e_1, e_2 can be eliminated from these equations in
the following way: Using the polytropic character of the gas and the
definition of $\gamma = 1 + R/C_v$, Eq. (3) can be written in the
form,

$$\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right)\frac{2\gamma}{\gamma-1} = v_1^2 - v_2^2 \tag{4}$$

From (1) and (2),

$$(p_2 - p_1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) = v_1^2 - v_2^2, \tag{5}$$

whence

$$\frac{\rho_1}{\rho_2} = \frac{p_1 + \frac{\gamma-1}{\gamma+1}p_2}{p_2 + \frac{\gamma-1}{\gamma+1}p_1} . \tag{6}$$

Now for a polytropic gas,

$$S_2 - S_1 = C_v \, \gamma \, \log \left(\frac{p_2}{p_1} \right)^{1/\gamma} \frac{\rho_1}{\rho_2}. \qquad (7)$$

Write $\quad p_2 / p_1 = 1 + \delta. \qquad (8)$

where δ is of the order of 0.1.

Substituting (6) and (8) into (7) and expanding it in a power series, it is found that the lowest order non-zero term in $S_2 - S_1$ is of the third order in δ and is given[*] by

$$S_2 - S_1 = R \, \frac{\gamma + 1}{12 \, \gamma^2} \, \delta^3. \qquad (9)$$

B. Attenuation of a Repeated Shock Wave.

Consider now a vibrating piston in a semi-infinite tube (see Figure (3)). Assume it to be vibrating for a sufficiently long time so
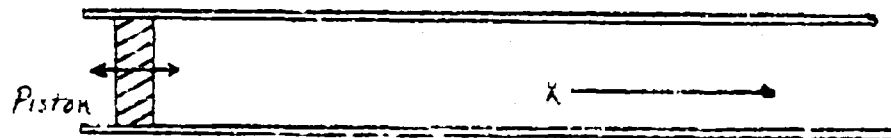

Fig. 3.

that the variation of hydrodynamical and thermodynamical quantities at any given point in the tube is truly periodic and that for all points the average mass flow is zero and the average temperature is uniform. Then, if the amplitude of the piston is such as to generate waves whose

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[*]See, e. g., p. 41 of Ref. 5.

pressure variation is of the order of one-tenth of the average pressure, after traveling a few wave lengths from the piston the fronts will have developed shocks. Assume that the pressure wave form at this distance is saw-tooth and can be pictured at a given time as in Figure 4. The amplitude of the discontinuity will decrease with X.
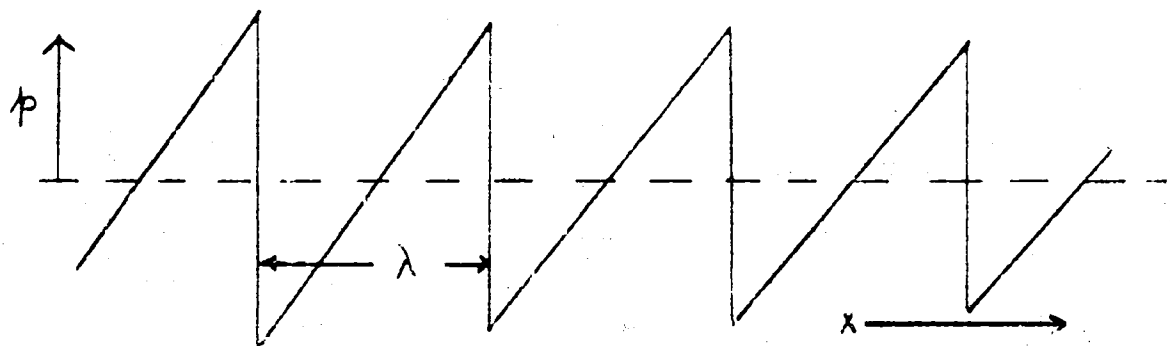
Figure (4)

At a given point, say $X = X_0$, the variation in pressure with time is indicated in Figure 5.
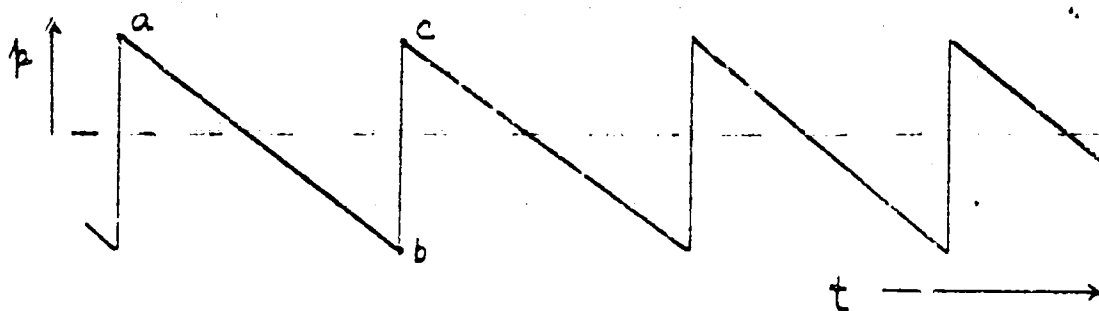
Figure (5)

Assume that the transition from a to b is isentropic — a completely reasonable assumption if the period is very much greater than the mean time between molecular collisions. On the other hand,

the change from $\underline{b}$ to $\underline{c}$ involves an increase in entropy as calculated in the previous section. For unit cross section of the wave the instantaneous rate at which entropy increases as a wave passes X_o can be obtained from (9) and is given by

$$\frac{dS}{dt} = \rho_o c_o (S_2 - S_1). \tag{10}$$

But

$$\frac{dS}{dt} = -\frac{1}{T} \frac{dE}{dt},$$

and hence,

$$\frac{dE}{dt} = -p_a c_o \frac{\gamma+1}{12\gamma^2} \delta^3, \tag{11}$$

where E is the mechanical energy of vibration per wave length per unit cross section and p_a is the average pressure. In writing (10), the velocity of the wave is taken as c_o, where c_o is the sound velocity at the mean temperature of the gas. This can be justified if it is noted that U is just equal to the mean value of $u + c$ on both sides of the shock to within first order terms. [*]

The energy E per wave length per unit cross section is

$$E = \frac{(p - p_a)^2_{r.m.s.}}{\rho c^2} \cdot \lambda. \tag{12}$$

With the assumption about the shape of the wave,

$$(p - p_a)^2_{r.m.s.} = \frac{p_a^2 \delta^2}{L^3/2} \int_0^{L/2} x^2 dx = \frac{p_a^2 \delta^2}{12},$$

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[*]See, e. g., p. 159 of Ref. 3.

and
$$E = \frac{p_a \, \delta^2 \lambda}{12 \, \gamma} \, . \tag{13}$$

Thus, using (11) and (13),

$$\frac{1}{E} \frac{dE}{dt} = - c_0 \frac{\gamma+1}{\gamma} \cdot \frac{\delta}{\lambda} \, , \tag{14}$$

and from this
$$\frac{1}{E} \frac{dE}{dx} = - \frac{\gamma+1}{\gamma} \cdot \frac{\delta}{\lambda} \, . \tag{15}$$

In terms of the rate of change of δ,

$$\frac{1}{\delta} \frac{d\delta}{dx} = - \frac{\gamma+1}{\gamma} \frac{\delta}{\lambda} \, . \tag{16}$$

If (16) is integrated an interesting relation ensues, namely,

$$\frac{1}{\delta} - \frac{1}{\delta_0} = \frac{\gamma+1}{2\gamma} \cdot \frac{x - x_0}{\lambda} \, . \tag{17}$$

C. Discussion.

It is seen from (17) that a plot of $\frac{1}{\delta}$ against the number

of wave lengths, $\frac{x - x_0}{\lambda}$ yields a straight line whose slope is wholly

determined by γ .   It is to be noted, as one would expect from the

nature of the initial assumptions, that the attenuation rate is independ-

ent of the attenuative process or the constants which characterize it.

The critical assumption is that the wave form becomes and remains

saw-tooth in character. Thus, any attenuative process which results

in the presence and stability of the saw-tooth wave form will lead to

an attenuation which is independent of the constants of the attenuative process.

It is worth emphasizing that the only entropy change which occurs in a wave length is associated with the shock itself, and that no entropy change occurs on the gradually sloping part of each wave. Consequently for waves which depart slightly from the assumed saw-tooth form (but which nevertheless possess a shock front and a gradually sloping part) of complete importance in calculating the rate of energy loss is the magnitude of the discontinuity. The nature of the gradually sloping part of the wave is involved only in the determination of the fraction of the wave energy the aforementioned power loss represents.

D. Comparison of Results with Related Theoretical Work.

In this section the results, equations (15), (16), and (17), will be discussed in the light of earlier related work by Dumond, Cohen, Panofsky and Deeds[4], and Fay[2].

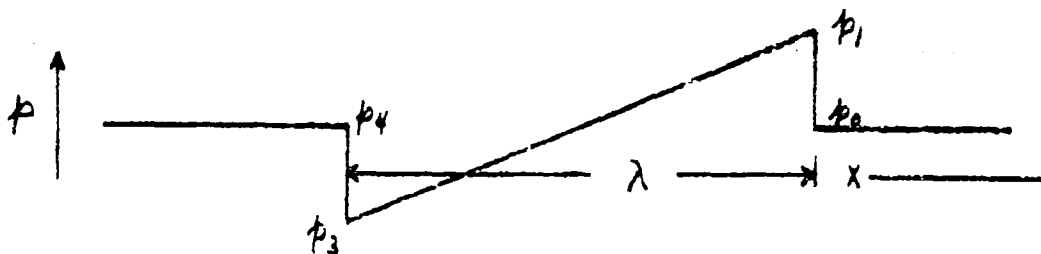Dumond et al.[4] derive the attenuation rate for the N wave pictured in Figure (6),

Figure (6)

where $\qquad \dfrac{p_1}{p_0} = \dfrac{p_3}{p_4} = 1 + \Delta.$

They find that

$$\frac{1}{E} \frac{dE}{dx} = \frac{\gamma + 1}{2\gamma} \frac{\Delta}{\lambda},$$ (18)

This result could also be obtained by the same methods used here. Applying (11) to the propagation of the two shock fronts of the N wave, one obtains by adding the two equal loss rates,

$$\frac{dE}{dx} = p_0 \frac{\gamma + 1}{6\gamma^2} \Delta^3,$$

and since $E = p_0 \Delta^2 \dfrac{\lambda}{3\gamma},$ it is seen that Eq. (18) follows.

As pointed out by Dumond et al., the propagation of the N wave is accompanied by a lengthening of the wave. That this is so is clear from the fact that the velocity, U, of the forward shock is given by the average of u + c on the two sides of the shock. It is thus equal to $c_0 + \dfrac{\gamma + 1}{4\gamma} u$. Similarly, the velocity of the rear shock is $c_0 - \dfrac{\gamma + 1}{4\gamma} u$. Thus,

$$\frac{d\lambda}{dt} = \frac{\gamma + 1}{2} u = \frac{\gamma + 1}{2\gamma} c_0 \Delta,$$

where the relation u = $p_a \Delta / p_0 c_0$ has been used. For the repeated shock wave, on the other hand, the velocity of consecutive fronts is the same to within terms of the first order in δ, and consequently

the wave length is constant.

It is worth while showing that the alternate hydrodynamic method Dumond et al. use to derive Eq. (18) may be very nicely applied to the repeated shock and yields Equations (15) and (16). Consider a linear pressure gradient in a sound wave represented by AB. Then, if O represents the point at which the particle velocity is zero,

Fig. 7.
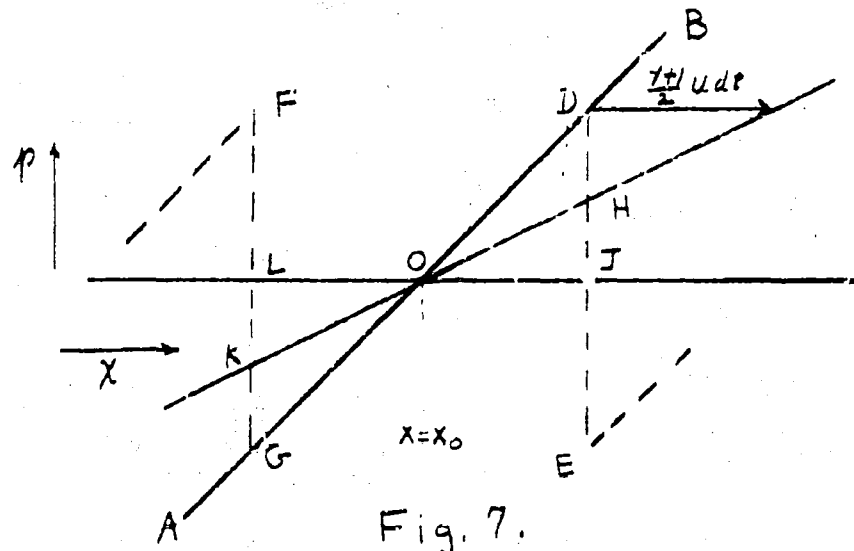
every given point on the line AB propagates away from this point with a velocity $(\gamma+1)u/2$, where u is the particle velocity at the given point in the wave. (The point O propagates with the velocity of sound, c_0 .) This is a result of solving the fundamental hydrodynamical equations. Thus, after a time interval, dt, the pressure curve would be described with respect to the displaced coordinate, $x_0 + c_0 dt$. However, if there are shock fronts DE and FG, so that GDH represents one wave of a repeated shock of the type being discussed, then DE will propagate with a velocity which is the mean value of c + u on both

sides of the shock which is equal to c_o . Thus, a wave which satisfies both requirements is one whose typical wave length is represented by LKHJ. (HJ is the half height of the discontinuity.) Now $DJ = p_a \frac{\delta}{2}$, and $DH = -d\left(p_a \frac{\delta}{2}\right)$. From the geometry

$$-\frac{d\left(p_a \frac{\delta}{2}\right)}{\frac{\gamma+1}{2} u \, dt} = \frac{p_a \frac{\delta}{2}}{\frac{\lambda}{2}} .$$

Whence, using $u = \frac{p_a \delta}{2 p_0 c_0}$, $c_o^2 = \frac{\gamma p_a}{p_0}$ and $c \, dt = dx$,

(16) is derived.

Fay's[2] result, Eq. (14)*, can be used to derive (16) of this paper. To show this, his equation is recast in terms of the variables used here.

$$\frac{p - p_0}{p_0} = \frac{2\gamma}{\gamma+1} \cdot \frac{4}{3} \cdot \frac{\mu \omega}{p_0 c_0^2} \sum_{m=1,2,}^{\infty} \frac{\sin m\left(\omega t - \frac{\omega x}{c_0}\right)}{\sinh m(\alpha_0 + \alpha x)}, \quad (18)$$

where $\alpha = \frac{2}{3} \frac{\mu \omega^2}{p_0 c_0^3}$,

$\frac{\mu}{p_0}$ = kinematic coefficient of viscosity,

$\omega = 2\pi \frac{c_0}{\lambda}$,

α_0 = constant.

As an example, determine the value of α_c , $(X = X_c)$, if the

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* Note that the factor 8 in Eq. (14) of Ref. 2 should be 2.

fundamental of the wave is to have a maximum value of $\frac{p - p_0}{p_0} = 0.05$,
$\omega = 200\pi$, $\gamma = 1.4$, $\mu/p_0 = 0.13$, $c_0 = 3.3 \cdot 10^4$.

It is found that $\sinh \alpha_0 \simeq 3.4 \cdot 10^{-6}$

Thus, for $n \lesssim 10^5$, the hyperbolic sine can be replaced by its argument and the amplitude of the harmonics is inversely proportional to the order of the harmonic, all harmonics being in phase. Moreover, since $\alpha = 0.96 \cdot 10^{-9}$, $\alpha x < \alpha_0$ for $x < 4$ meters. With conventional measuring instruments the wave will appear to retain its saw-tooth character for many wave lengths of travel. For example, if a microphone's response drops off seriously above 10 kc (the 100th harmonic of the 100 c.p.s. wave), the calculation indicates that the distance of propagation before a significant departure from saw-tooth character would be observed is approximately 15,000 meters (assuming, of course, plane wave propagation).

To continue with the original purpose, suppose

$$\left(\frac{p - p_0}{p_0}\right)_{max} \sim 0.03$$

Then $\frac{p - p_0}{p_0} \simeq \frac{\delta}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\left(\omega t - \frac{\omega x}{c_0}\right)}{n}$,

where δ is given by Eq. (8).

$$\frac{\pi}{\delta} = \frac{\gamma+1}{2\gamma} \frac{\alpha_0 + \alpha x}{\frac{4}{3} \frac{\mu \omega}{\rho_0 c_0^2}} \ ,$$

$$d\left(\frac{1}{\delta}\right) = \frac{\gamma+1}{2\gamma} \cdot \frac{dx}{\lambda} \ ,$$

which checks Eq. (18).

It is perhaps worth emphasizing a point which has been made by Fay — if one adopts the approach, which most people in acoustics think of naturally, of reducing a wave to its Fourier components, then the attenuation of a saw-tooth wave occurs principally in its high frequency components. The procedure by which a saw-tooth wave maintains its form is by the low frequency components feeding energy into the higher harmonics and the higher harmonics losing vibrational energy through the action of viscosity. Thus, in Eq. (19) the term $m \alpha x$ in the numerator is associated with attenuation, and a large value of $m \alpha x$ (e. g., $m \gg 10^5$ in the example taken) is evidence of a large rate of attenuation.

Another way of looking at this is to calculate the rate of energy loss of a saw-tooth wave, which maintains its shape, by assuming that the total rate of energy loss is the sum of those due to the harmonics, assuming each harmonic is attenuated at a fractional rate which is given by infinitesimal amplitude acoustics. Let E_n be the vibrational energy density associated with the nth harmonic, and E_t be the total energy density. Then

$$\frac{d E_t}{d x} = \sum_{n=1,2}^{\infty} \frac{d E_n}{d x} .$$

For a saw-tooth wave $\qquad E_m = \frac{E_1}{m^2} .$

With our assumption,

$$E_m = E_{mo} \, e^{-2\alpha_1 m^2 x} ,$$

where α_1 is the value of α (defined below Eq. (18)) for the funda-mental.

$$\frac{1}{E_t} \frac{d E_t}{d x} = \frac{1}{E_t} \sum_{n=1,2} \frac{d E_m}{d x} = - \frac{\sum_{m=1}^{\infty} \frac{E_1}{m^2} 2\alpha_1 m^2}{E_1 \sum_{m=1}^{\infty} \frac{1}{m^2}} .$$

The summation in the numerator does not converge. On the other hand, if the assumption is made that the harmonics of order greater than $m = N$ are absent, then,

$$\frac{1}{E_t} \frac{d E_t}{d x} = \frac{N \, 2\alpha_1}{1.58}$$

Suppose now a determination is made of the number of har-monics N which is necessary to give the rate of attenuation given by Eq. (16).

$$N = \frac{1.58}{2\alpha_1} \cdot \frac{\gamma+1}{\gamma} \cdot \frac{\delta}{\lambda} .$$

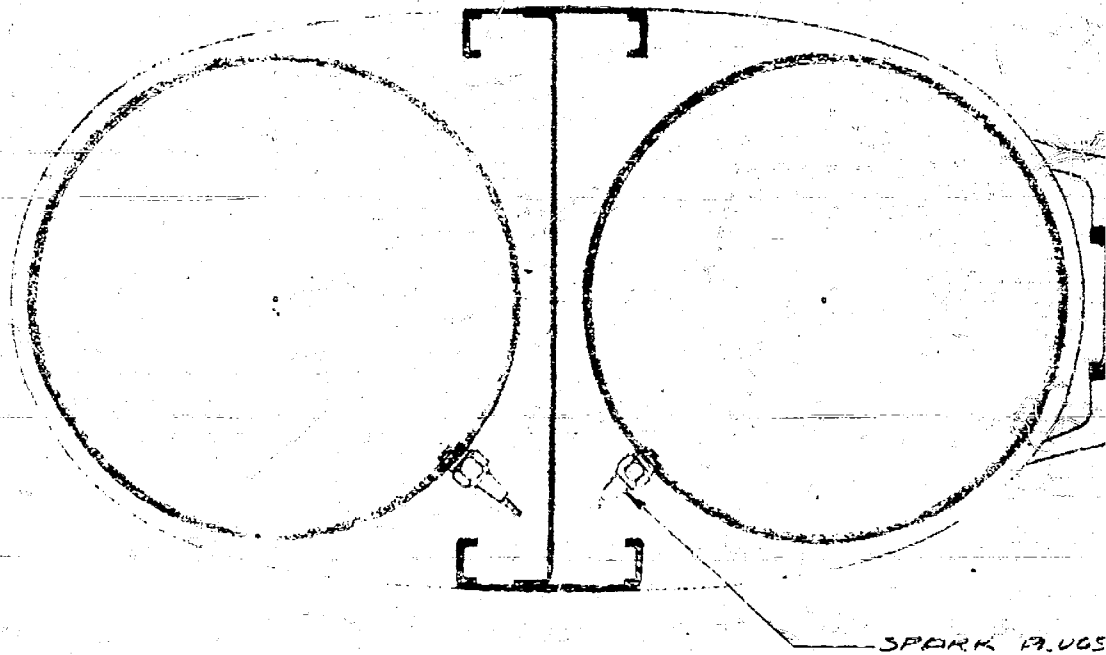For a fundamental of 100 c.p.s. and $\delta = 0.03$,

$$N = 1.3 \cdot 10^5 .$$

Since there are more than ten times as many harmonics between 1 and 13 megacycles as below 1 megacycle, this calculation would indicate that most of the energy lost by viscous action occurs for components above 1 megacycle. While the details of this calculation cannot be correct, this last conclusion is correct with better than order of magnitude accuracy.

References

1. Rayleigh, "Theory of Sound", Vol. II, MacMillan & Co.,
    New York, 1896.

2. R. D. Fay, Journal of the Acoustical Society of America,
    2, 222 (1931).

3. See e. g., Courant and Friedericks, "Supersonic Flow
    and Shock Wave Phenomena", Interscience Publishers,
    Inc., New York, 1948.

4. Dumond, Cohen, Panofsky and Deeds, Journal of the
    Acoustical Society of America, 18, 97, 1946.

5. Liepmann and Puckett, "Introduction to Aerodynamics of
    a Compressible Fluid", John Wiley and Sons, New York,
    1947.

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SECTION C-C

SPARK PLUGS

SECTION-BB

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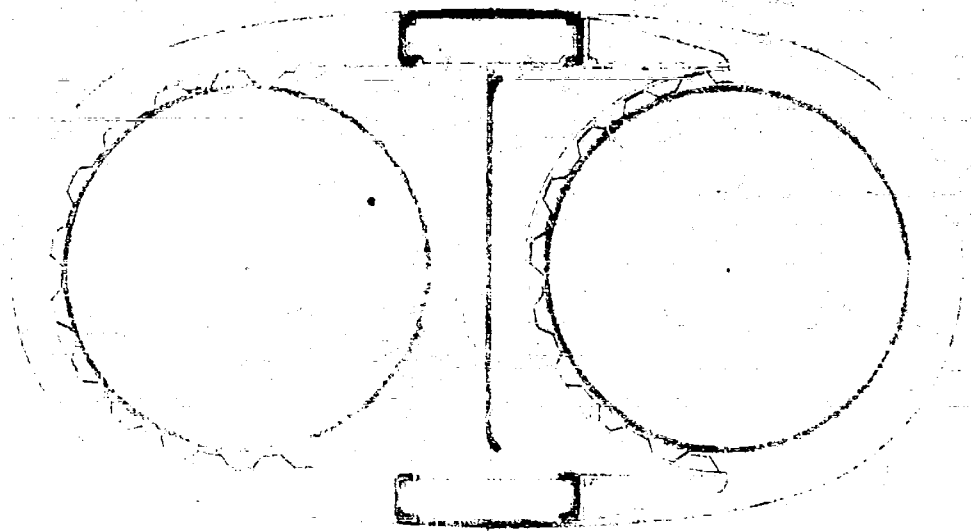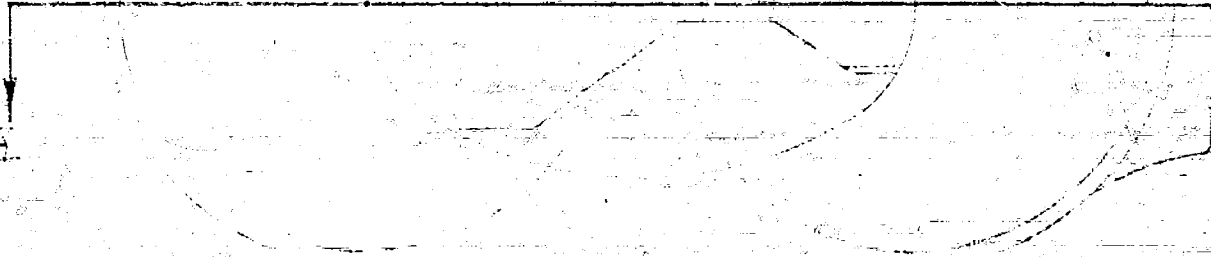
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STARTING AIR
MANIFOLD & NOZZLE

5S — FOR STARTING ONLY

IIB

S.F.

AIR ENTERS COOLING
DUCT HERE

COOLING &
AUXILIARY AIR
INLETS

SECTION III-III

7

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SECTION $\overline{II}$ II

II — III

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E. 3

PRELIM. DESIGN & INTERNAL
COMPLETE TWIN SONIC POWER
FOR ROTARY WING UNITS

PROJECT ENGINEER   SCALES   DRAWING No.

CHECKED BY